Name:______________________

MA 8633 Section 01 | *Practice Exam 1* | *November 19, 2019*

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work, answer or there is no justification to a solution you will receive little or no credit!

**1**. Let $A \subseteq \mathbb{R}$ be a measurable set. Define the set

$$B = A \cup -A \quad \text{where} \quad -A := \{-x : x \in A\} \ .$$

Prove that $B$ is measurable and $mA \leq mB \leq 2mA$.

**2**. Let $A$ and $B$ be two compact subsets of $\mathbb{R}$. Prove that the set

$$A + B = \{x + y : x \in A, y \in B\}$$

is closed and that

$$m(A + B) \geq m(A) + m(B) \ .$$

**3**. Let $\{f_n\}$ be a sequence of measurable functions on $E$ that converges to the real-valued $f$ pointwise on $E$. Prove that

$$E = \bigcup_{k=1}^{\infty} E_k$$

where for each index $k$, $E_k$ is measurable, $\{f_n\}$ converges uniformly to $f$ on each $E_k$ if $k > 1$, and $m(E_1) = 0$.

**4**. Let $f$ be a Lipschitz function on $[0, 1]$. Prove that $f$ is differentiable almost everywhere on $[0, 1]$.

**5**. Let $\{f_n\}$ be a sequence of measurable functions on $(0, 1)$. Prove that

$$E = \{x \in (0, 1) : \{f_n(x)\} \text{ is a convergent sequence}\}$$

is measurable.

**6**. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f^{-1}(c)$ is measurable for every number $c$. Is $f$ necessarily measurable? Be sure to justify.

**7**. Prove the following limit exists and compute it:

$$\lim_{n\to\infty} \int_0^1 \frac{e^{-x}\cos x}{nx^2 + \frac{1}{n}}\, dx \; .$$

**8**. Prove or disprove that the Bounded Convergence Theorem holds for the Riemann integral.